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Diffraction of a Plane Electromagnetic Wave at a Schwarzschild Black Hole

E. HERLT and H. STEPHANI

Abteilung Relativistische Physik, Sektion Physik, Friedrich-Schiller-Universität Jena, DDR

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Abstract

Using the technique of Debye potentials a rigorous solution of the diffraction problem is given as a superposition of an incident wave, strongly connected with the Coulomb scattering wave function, and a scattered wave, which is purely outgoing for large distances. The solution fulfils the boundary conditions to be the light of a very distant star and to be purely ingoing at the Schwarzschild horizon. The phase shifts of the partial waves are evaluated in the WBK approximation.

1. *Introduction*

The deflection of light rays coming from a distant star and passing the sun was one of the first problems solved in the history of general relativity, and it is nowadays treated in every textbook. Our goal is to give a full wave theoretical treatment of this problem. Because in the most interesting cases the Schwarzschild radius will be large compared with the wavelength of the electromagnetic field, the results will not differ significantly from those of geometrical optics, with the exception of the regions in the geometrical shadow or the regions where each point is passed by more than one ray.

This paper starts with a short account of notations, the technique of Debye potentials and some properties of generalised plane waves. The main results are in Section 3 the phase shifts (3.4), (3.8) and (3.10) of the partial waves; in Section 4 the asymptotic form (4.7) of the incident wave, in Section 5 the evaluation (5.8) of the partial wave amplitudes, and in Section 6 the representation (6.6) of the Debye potential of the incident wave in terms of the Coulomb scattering function.

2. *Metric, Debye Potentials and Generalised Plane Wave*

It is convenient to use besides the Schwarzschild metric

$$d\bar{s}^2 = r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) + \frac{r}{r-1}dr^2 - \frac{r-1}{r}dt^2 \tag{2.1}$$

the metric

$$ds^2 = \frac{r^3}{r-1}(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) + dv^2 - dt^2, \qquad v = r + \ln(r-1) \tag{2.2}$$

which is conformally equivalent to (2.1). Distances are measured in units of the Schwarzschild radius.

To get the general solution of Maxwell equations in this background, one has to solve the Debye equation (Stephani, 1974)

$$\frac{r-1}{r^3}\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial\pi}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2\pi}{\partial\varphi^2}\right] + \frac{\partial^2\pi}{\partial v^2} - \frac{\partial^2\pi}{\partial t^2} = 0 \tag{2.3}$$

and to insert two general solutions π and ϕ into

$$\begin{aligned} A_a &= \pi_{,n}(u^n v_a - v^n u_a) + \epsilon_a{}^{bpq}\phi_{,b}v_p u_q \\ v^a &= (0,0,1,0) \qquad u^a = (0,0,0,1) \end{aligned} \tag{2.4}$$

Formula (2.4) holds in the metric (2.2), but A_a is the four potential of an arbitrary field in the physical space-time (2.1) too:

$$F_{\bar{a}\bar{b}} = A_{b,a} - A_{a,b} \tag{2.5}$$

In flat space-time a plane, monochromatic, linearly polarised electromagnetic wave travelling in the z-direction, with the non-zero components

$$E_x = H_y = e^{i\omega(z-t)} \tag{2.6}$$

belongs to Debye potentials connected by

$$\pi = P(r,\vartheta)\cos\varphi\, e^{-i\omega t}, \qquad \phi = -P(r,\vartheta)\sin\varphi\, e^{-i\omega t} \tag{2.7}$$

(Hönl *et al.*, 1961). As the symmetry with respect to rotations around the z-axis is not altered by the gravitational field (2.7) is valid for our problem too. Change of polarisation changes the relation (2.7) between π and ϕ, but does not affect P. Because of the Debye equation (2.3) P has to fulfil

$$\frac{r-1}{r^3}\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial P}{\partial\vartheta} - \frac{P}{\sin^2\vartheta}\right] + \frac{\partial^2 P}{\partial v^2} + \omega^2 P = 0 \tag{2.8}$$

and the components of the electromagnetic field are in terms of P

$$F_{\vartheta\varphi} = \alpha \sin\varphi\, e^{-i\omega t} \qquad F_{\varphi t} = \delta \sin\varphi\, e^{-i\omega t}$$

$$F_{\vartheta\vartheta} = \delta \frac{\cos\varphi}{\sin\vartheta} e^{-i\omega t} \qquad F_{\vartheta t} = \alpha \frac{r-1}{r^3} \frac{\cos\varphi}{\sin\vartheta} e^{-i\omega t} \tag{2.9}$$

$$F_{\vartheta t} = \beta \frac{\cos\varphi}{\sin\vartheta} e^{-i\omega t} \qquad F_{\varphi v} = \beta \sin\varphi\, e^{-i\omega t}$$

with

$$\alpha = -\sin\vartheta \frac{\partial}{\partial\vartheta} \frac{1}{\sin\vartheta} \frac{\partial}{\partial v} (\sin\vartheta P)$$

$$\beta = \sin\vartheta \frac{\partial}{\partial\vartheta} \frac{\partial}{\partial v} P + i\omega P \tag{2.10}$$

$$\delta = -i\omega \sin\vartheta \frac{\partial P}{\partial\vartheta} - \frac{\partial P}{\partial v}$$

The general solution of the reduced Debye equation (2.8) is

$$P(r, \vartheta) = \sum_{n=1}^{\infty} D_n R_n(r) P_n^1(\cos\vartheta) \tag{2.11}$$

Here $P_n^1(\cos\vartheta)$ are the Legendre functions, D_n arbitrary constants and R_n solutions of the radial equation

$$\frac{d^2 R_n}{dv^2} + \omega^2 [1 - a^2 V(v)] R_n = 0$$

$$a^2 = \frac{n(n+1)}{\omega^2}, \qquad V(v) = \frac{r-1}{r^3} \tag{2.12}$$

To solve our diffraction problem means to find a solution of (2.8) which (a) corresponds for $\vartheta \approx \pi$, $r \to \infty (z \to -\infty)$ to an incident plane wave and (b) is purely ingoing at $r = 1$ ($v = -\infty$). In terms of (2.11) condition (a) will give us the D_n, whereas condition (b) singles out one of the two independent solutions of the radial equation.

3. *The Radial Equation*

The radial equation (2.12) was studied by several authors interested in electromagnetic radiation fields in the Schwarzschild background; an incomplete list is Mo & Papas (1970), Price (1972), Misner *et al.* (1972), Breuer *et al.* (1973), Ruffini *et al.* (1972), Stephani & Herlt (1973). Matzner (1968) dealt with a similar equation valid for scalar waves.

Written in the variable r the radial equation reads

$$\frac{d^2R_n}{dr^2}+\frac{1}{r(r-1)}\frac{dR_n}{dr}+\left[\frac{\omega^2r^2}{r-1}-\frac{n(n+1)}{r(r-1)}\right]R_n=0 \tag{3.1}$$

According to our boundary condition at $r = 1$, we have to take a solution R_n which is a purely ingoing wave at $v = -\infty$. This is fulfilled by a wave $e^{-i\omega\vartheta}$, coming from $v = +\infty$ and being reflected or transmitted by the potential $V(v)$.

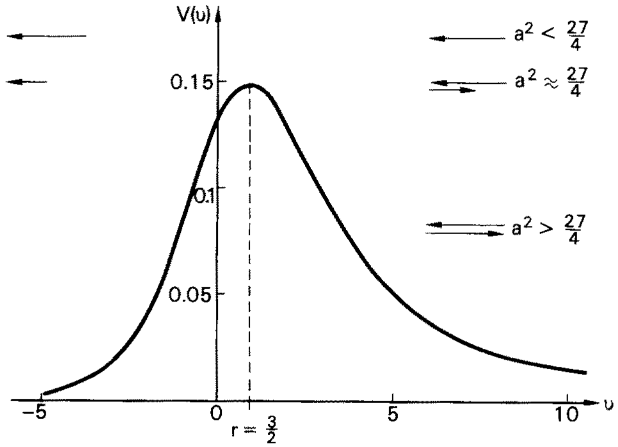

Figure 1.—Radial waves reflected and/or transmitted by the potential $V(v)$.

No exact solution is known for $n \neq 0$, and we have to use approximation methods. The frequency ω—which in our units is the number of wavelengths per Schwarzschild radius—is large for most of the applications, and so the WKB-approximation is applicable:

$$R_n(v)=A(v)\,e^{\pm\, i\omega S(v)}$$

$$S(v)=\int^{v}\sqrt{[1-a^2V(v)]}\,dv, \qquad A(v)=[1-a^2V(v)]^{-1/4} \tag{3.2}$$

The result will be simple and of practical use only in the region near the black hole ($r \to 1, v \to -\infty$) or for large distances ($r \to \infty, v \to +\infty$): in both cases $V(v)$ tends to zero, and all information is contained in the phase shift of the in- and out-going waves. We now will calculate these phase shifts for the three cases physically significant.

Case I: $a^2 < 27/4$

These partial waves with small angular momentum correspond to light rays entering the black hole. A wave $e^{-i\omega v}$ passes from $v = +\infty$ to $v = -\infty$ and gets the total phase shift $\omega\Delta_1 S$:

$$v = +\infty: \quad R_n = e^{-i\omega v}$$

$$v = -\infty: \quad R_n = e^{-i\omega(v - \Delta_1 S)} \tag{3.3}$$

$$\Delta_1 S = \int_{-\infty}^{+\infty} \{1 - \sqrt{[1 - a^2 V(v)]}\}\, dv = \int_1^{\infty} \left[1 - \sqrt{\left(1 - a^2 \frac{r-1}{r^3}\right)}\right] \frac{r}{r-1}\, dr$$

We evaluate this elliptic integral by expanding the square root in a series with respect to a^2 and integrating term by term:

$$\begin{aligned} \Delta_1 S &= \sum_{m=1}^{\infty} a^{2m} (-1)^m \binom{\frac{1}{2}}{m} \sum_{l=0}^{m} (-1)^1 \binom{m-1}{l} \frac{1}{2m+l-1} \\ &= \tfrac{1}{2} a^2 + \tfrac{1}{96} a^4 + \tfrac{1}{1680} a^6 + \tfrac{1}{21504} a^8 + \tfrac{7}{256}\, \tfrac{1}{16435} a^{10} + \cdots \end{aligned} \tag{3.4}$$

The accuracy of this series can be tested by comparing the value for $a^2 = 27/4$ with the exact value 4,4390295 . . . (in this special case the integral is not elliptic).

Case II: $a^2 > 27/4$

These partial waves with large angular momentum correspond to light rays passing outside $r = 3/2$. An ingoing wave $e^{-i\omega v}$ is completely reflected and gets the total phase shift $\omega\Delta_2 S$ (including the typical WBK shift $\pi/2$ near the classical turning point v_0)

$$v = +\infty: R_n = e^{-i\omega v} - e^{i\omega(v - \Delta_2 S)}$$

$$\Delta_2 S = 2 \int_{v_0}^{\infty} \{1 - \sqrt{[1 - a^2 V(v)]}\}\, dv + 2v_0 - \frac{\pi}{2\omega} \tag{3.5}$$

$$= 2I + 2v_0 - \frac{\pi}{2\omega}$$

There are two ways of evaluating this elliptic integral in the form of a series. Both start from

$$I = \int_1^{\infty} \left\{1 - \sqrt{\left[1 - \frac{r_0 w - 1}{(r_0 - 1) w^3}\right]}\right\} \frac{r_0^2 w}{r_0 w - 1}\, dw \tag{3.6}$$

which follows from (3.5) after substitution of $a^2 = r_0^3/(r_0 - 1)$ and $r = r_0 w$.

The first is to write I as

$$I = r_0 \int_1^{\infty} \left\{1 - \sqrt{\left(\frac{w^2 - 1}{w^2}\right)} \cdot \sqrt{\left[1 - \frac{1}{w(w+1)(r_0 - 1)}\right]}\right\} \left(1 + \frac{1}{r_0 w - 1}\right) dw \tag{3.7}$$

to expand the second square root, to integrate term by term, and to make a second expansion in powers of a. The result of this rather lengthy calculation is

$$I = \left(\frac{\pi}{2} - 1\right) a + 1 - \ln 2 + \frac{15}{8}\left(1 - \frac{\pi}{4}\right)\frac{1}{a} + \frac{2}{3}\frac{1}{a^2} + \cdots \tag{3.8}$$

This series is useful for large values of a, but converges rather badly near $a^2 = 27/4$ and not at all at this point.

The second way is to expand the square root in (3.6):

$$I = \sum_{m=1}^{\infty} I_m = \sum_{m=1}^{\infty} \frac{1.1.3\ldots(2m-3)}{1.2.4\ldots 2m} \int_1^{\infty} \frac{r_0^2 w}{r_0 w - 1}\left[\frac{r_0 w - 1}{(r_0 - 1)w^3}\right]^m dw \tag{3.9}$$

The final result will be a series of the form

$$I = \frac{r_0^2}{r_0 - 1}\left[B_1 + \frac{B_2}{r_0 - 1} + \frac{B_3}{(r_0 - 1)^2} + \cdots\right] \tag{3.10a}$$

but unfortunately each of the B_m is an infinite sum of contributions of different I_m. But by substituting $a^2 = r_0^3/(r_0 - 1)$ into (3.10a) and reordering we can express $B_1 \ldots B_4$ in terms of the coefficients of (3.8) and guess the values of B_5 and B_6:

$$B_1 = \frac{\pi}{2} - 1 \qquad B_4 = \tfrac{95}{64}\pi - \tfrac{31}{12} - 3\ln 2$$

$$B_2 = \tfrac{3}{2} - \ln 2 - \frac{\pi}{4} \qquad B_5 \approx 0{,}000\,125 \tag{3.10b}$$

$$B_3 = 2\ln 2 - \tfrac{1}{2} - \tfrac{9}{32}\pi \qquad B_6 \approx 0{,}000\,05$$

The series (3.10) is convergent for $a^2 \geqslant 27/4$; the exact value for $a^2 = 27/4$ is $I = 2{,}8558$.

Case III: $a^2 \approx 27/4$

For classical turning points near the summit of the potential $V(v)$ (for rays grazing $r = 3/2$) the WBK approximation fails. Following Ford *et al.* (1959), near the summit we take the exact solution of the radial equation with a parabolic potential, and combine it with the WBK solution outside this region. We will not go into the details, but just give the main results.

Using the notation $N(N + 1) = (27/4)\omega^2$, $n = N + p$, the properties of the reflected and transmitted waves can be given in terms of the parameter $\epsilon = -1{,}208p$.

The ratio of the amplitudes of the reflected and transmitted waves, taken at points of the same $V(v)$, is

$$\left|\frac{R_{n\,(\text{refl.})}}{R_n(\text{tr.})}\right| = e^{-\pi\epsilon} \tag{3.11}$$

This shows, that at most four partial waves give a practically non-zero ratio and are, therefore, not included in Case I or II of this section. For large negative v the transmitted wave is

$$R_n = \frac{e^{-i\omega v}\, e^{\pi\epsilon/4}}{[e^{\pi\epsilon/2} + e^{-3\pi\epsilon/2}]^{1/2}}$$

$$\times \exp\,[-i(4{,}48637\omega - \epsilon(\ln\omega - 0{,}49742 - 0{,}06766\epsilon^2))] \tag{3.12}$$

For large positive v the incident plus reflected waves behave like

$$R_n = e^{-i\omega v} - \frac{e^{i\omega v}\, e^{-3\pi/4}}{[e^{\pi\epsilon/2} + e^{-3\pi\epsilon/2}]^{1/2}} \tag{3.13}$$

$$\times \exp\,[-i(7{,}37182\omega - \frac{\pi}{2} - \epsilon(\ln\omega - 0{,}1066 - 0{,}06767\epsilon^2))]$$

We close this section on properties of the radial functions R_n with the remark that one can obtain approximation solutions in terms of Bessel and Weber-Hermite functions by using approximations for the potential $V(v)$ for finite values of v.

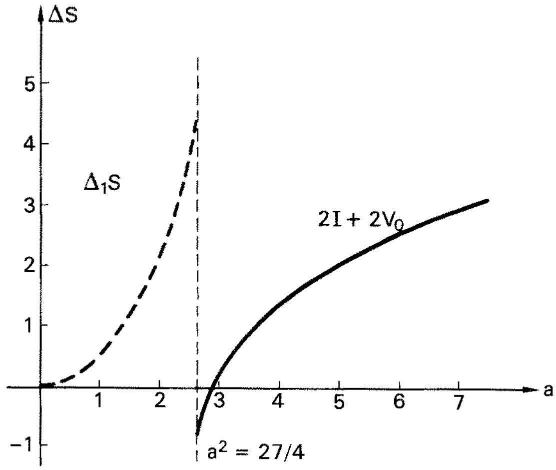

Figure 2.–Phase shifts of the radial waves.

------- $a^2 < 27/4 \quad v \to -\infty$

——— $a^2 > 27/4 \quad v \to +\infty$

4. *The Asymptotic Behaviour of the Incident Wave*

In the absence of the black hole the incident wave would be a plane wave (2.6). It is known from Coulomb scattering that a far-reaching potential proportional to r^{-1} alters the incoming plane wave even at infinity. We will now discuss this problem for the gravitational case.

The condition that the light comes from a very distant star implies that the light rays (null geodesics) are parallel for $r \to \infty$, $\vartheta \approx \pi$ $(z \to -\infty)$. The equation of null geodesics in the Schwarzschild background (Darwin, 1959)

$$\left(\frac{du}{d\vartheta}\right)^2 = u^3 - u^2 + \frac{1}{B^2}, \qquad u = \frac{1}{r} \tag{4.1}$$

gives the requested congruence (up to terms linear in the gravitational constant) in the form

$$B(u, \vartheta) = \frac{\sin\vartheta}{u} + \frac{1}{2}\frac{(1+\cos\vartheta)^2}{\sin\vartheta} = \text{const.} \tag{4.2}$$

The phase $W(u, \vartheta)$ of the incident wave is a solution of the Hamilton-Jacobi (Eiconal-) equation

$$\left(\frac{\partial W}{\partial\vartheta}\right)^2 u^2(1-u) + \left(\frac{\partial W}{\partial u}\right)^2 u^4(1-u)^2 = 1 \tag{4.3}$$

and the lines of constant phase should be orthogonal to the light rays

$$W_{,u}B^{,u} + W_{,\vartheta}B^{,\vartheta} = 0 \tag{4.4}$$

The solution

$$W(r, \vartheta) = (r - \tfrac{1}{2})\cos\vartheta - \ln r(1 - \cos\vartheta) \tag{4.5}$$

of the system (4.3)-(4.4) differs from the flat space solution $W_0 = r\cos\vartheta$ even at large r.

On this background of geometrical optics we try the ansatz

$$A_a = \hat{A}_a(r, \vartheta)\, e^{i\omega(W-t)} \tag{4.6}$$

for the four potential. Maxwell equations then give (again in the linear approximation)

$$A_\vartheta \approx \frac{r}{i\omega}\cos\varphi\cos\vartheta\left(1 - \frac{1}{2r\cos\vartheta}\right) e^{i\omega(W-t)}$$

$$A_\varphi \approx -\frac{r}{i\omega}\sin\varphi\sin\vartheta\, e^{i\omega(W-t)} \tag{4.7}$$

$$A_v \approx \frac{1}{i\omega} \cos\varphi \sin\vartheta \left(1 + \frac{2\cos\vartheta - 1}{2r(1-\cos\vartheta)}\right) e^{i\omega(W-t)}$$

$$A_t \approx \frac{1}{i\omega} \cos\varphi \sin\vartheta \frac{1}{2r(1-\cos\vartheta)} e^{i\omega(W-t)}$$

This result is confirmed by calculations of Herlt (to be published), who directly computed the field of an oscillating dipole at infinity. (4.7) is not within the gauge used in (2.4).

The field component F_{vt} is of particular interest; due to (2.9) it is rather simple written in terms of (2.11):

$$F_{vt} = \cos\varphi \sum_{n=1}^{\infty} \frac{n(n+1)(r-1)}{r^3} D_n R_n(r) P_n^1(\cos\vartheta)\, e^{-i\omega t} \tag{4.8}$$

For the incident wave (4.7) it has the form

$$F_{vt} \approx \cos\varphi \sin\vartheta \left(1 - \frac{3}{2r} + \frac{1}{r(1-\cos\vartheta)} + \frac{i\cos\vartheta}{2r^2(1-\cos\vartheta)}\right) e^{i\omega(W-t)} \tag{4.9}$$

Neglecting the last term in the bracket, we can write this as

$$F_{vt} \approx \frac{i\cos\varphi}{\omega(r+1)} e^{-i\omega t} \frac{\partial}{\partial\vartheta} e^{i\omega W(r,\vartheta)} \tag{4.10}$$

5. *Evaluation of the Partial Wave Amplitudes D_n*

As mentioned before, the properties of the incident wave should give us the partial wave amplitudes D_n. At first sight it seems to be rather easy to extract them from (4.8) and (4.9): one should expand (4.9) in a series with respect to $P_n^1(\cos\vartheta)$ and compare the coefficients with those of (4.8) for large r. But this does not work, because the fields (4.7) and (4.9) are singular at $\vartheta = 0$ and cannot be written in terms of $P_n^1(\cos\vartheta)$ at all.

Our first problem, therefore, is to find an electromagnetic field, which for $r \to \infty$, $\vartheta \approx \pi$ has the same asymptotic behaviour as the incident field given in (4.9), but which is regular everywhere, for all values of ϑ and v. This field can be constructed using the solution

$$\psi = \Gamma(1 - i\omega)\, e^{\pi\omega/2} \exp[i\omega(r - \tfrac{1}{2})\cos\vartheta] F[i\omega | 1 | i\omega(r - \tfrac{1}{2})(1 - \cos\vartheta)] \tag{5.1}$$

of the quantum mechanical Coulomb scattering problem (Messiah, 1961, p. 422). The parameters have been adjusted for our purposes; $F[\alpha|\beta|z]$ is the confluent hypergeometric series, the regular solution of

$$z\frac{d^2F}{dz^2} + (\beta - z)\frac{dF}{dz} - \alpha F = 0 \tag{5.2}$$

For $r \to \infty$ this wave has the asymptotic form

$$\psi \approx e^{-i\omega \ln \omega}\, e^{i\omega W(r, \vartheta)} \tag{5.3}$$

Formulae (5.3) and (4.10) indicate that

$$\overset{\circ}{F}_{vt} = \frac{i \cos \varphi\, e^{-i\omega t}}{\omega(r+1)}\, e^{i\omega \ln \omega} \frac{\partial}{\partial \vartheta} \psi \tag{5.4}$$

is a suitable definition of the wanted regular field.

Now we can get the D_n by a straightforward calculation. We insert into (5.4) the expansion

$$\psi = \sum_{n=0}^{\infty} \frac{2n+1}{\omega(r-\frac{1}{2})} i^n\, e^{i\sigma n} F_n[-\omega, (r-\tfrac{1}{2})\omega] P_n(\cos \vartheta) \tag{5.5}$$

$$\sigma_n = \arg \Gamma(1 + n - i\omega)$$

of the Coulomb wave function and get

$$\overset{\circ}{F}_{vt} = -\cos \varphi\, e^{-i\omega t} \sum_{n=1}^{\infty} \frac{i^{n+1}\, e^{i\omega \ln \omega}}{(r-\frac{1}{2})(r+1)\omega^2} (2n+1)\, e^{i\sigma_n}$$

$$\times F_n[-\omega, \omega(r-\tfrac{1}{2})] P_n^1(\cos \vartheta) \tag{5.6}$$

The exact solution (4.8) should differ from the incident wave (5.6) only by outgoing waves, the amplitudes of the ingoing waves should be equal. Remembering

$$F_n \sim \sin [\omega(r-\tfrac{1}{2}) + \omega \ln 2\omega r - \frac{n}{2}\pi + \sigma_n] \tag{5.7}$$

we get

$$D_n = \frac{(-1)^n}{2\omega^2} \frac{2n+1}{n(n+1)}\, e^{i\omega(1/2 - \ln 2)} \tag{5.8}$$

We could have guessed this result, because apart from the unimportant factor exp.$(i\omega(1/2 - \ln 2))$ these are exactly the amplitudes of the ingoing waves in flat space time (Hönl, 1961): formula (5.8) only says, that the black hole does not alter the amplitudes of the incoming waves, i.e. it does not alter the structure of the source.

6. *The 'Incident' and 'Scattered' Waves*

We are now left with the problem of summing up

$$P(r, \vartheta) = \sum_{n=1}^{\infty} D_n R_n(r) P_n^1(\cos \vartheta) \tag{6.1}$$

Our goal is to split (6.1) into the incident wave $\overset{\circ}{P}$, which incorporates the regular field defined in Section 5, and the scattered wave, which can be treated with methods of the usual scattering theory.

We start from (5.6), writing it now as

$$\overset{\circ}{F}_{vt} = \cos\varphi\, e^{-i\omega t} \sum_{n=1}^{\infty} \frac{n(n+1)(r-1)}{r^3} D_n \overset{\circ}{R}_n(r) P_n^1(\cos\vartheta) \qquad (6.2)$$

which implies

$$\overset{\circ}{R}_n = \frac{2(-i)^{n+1} r^3}{(r^2-1)(r-\frac{1}{2})} e^{i\omega(\ln 2\omega - 1/2)}\, e^{i\sigma_n} F_n[-\omega, \omega(r-\tfrac{1}{2})] \qquad (6.3)$$

As incident wave we define

$$\overset{\circ}{P} = \sum_{n=1}^{\infty} D_n \overset{\circ}{R}_n(r) P_n^1(\cos\vartheta) \qquad (6.3)$$

which just gives (6.2) if we apply (2.9). By integrating the equation

$$\begin{aligned}\overset{\circ}{F}_{vt} &= -\frac{r-1}{r^3}\cos\varphi\, e^{-i\omega t} \frac{\partial}{\partial\vartheta}\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}(\overset{\circ}{P}\sin\vartheta) \\ &= \frac{i\cos\varphi\, e^{-i\omega t}}{\omega(r+1)} e^{i\omega\ln\omega} \frac{\partial}{\partial\vartheta}\psi \end{aligned} \qquad (6.5)$$

which follows from (2.9) and (5.4), we get

$$\begin{aligned}\overset{\circ}{P}(r,\vartheta) = \frac{i\, e^{i\omega\ln\omega}}{\omega(r^2-1)} \frac{r^3}{\sin\vartheta} \Bigg[\int_{-1}^{\cos\vartheta} \psi(r,\vartheta) d\cos\vartheta \\ - (1+\cos\vartheta) \frac{e^{i\sigma_0}}{\omega(r-\frac{1}{2})} F_0[-\omega, \omega(r-\tfrac{1}{2})] \Bigg] \end{aligned} \qquad (6.6)$$

giving us the Debye potential of the incident wave in terms of the Coulomb wave function ψ, i.e. essentially in terms of the confluent hypergeometric series.

The exact solution $P(r, \vartheta)$ of our scattering problem can now be written in the form

$$P(r,\vartheta) = \overset{\circ}{P}(r,\vartheta) + \sum_{n=1}^{\infty} D_n P_n^1(\cos\vartheta)[R_n(r) - \overset{\circ}{R}_n(r)] = \overset{\circ}{P} + \overset{s}{P} \qquad (6.7)$$

In order to recognise the properties of the second term, which we will call the scattered wave, we make the connection between R_n and $\overset{o}{R}_n$ more obvious by writing down their differential equations (. . . indicate terms of higher order in r^{-1})

$$0=\frac{d^2R_n}{dr^2}+\left(\frac{1}{r^2}+\ldots\right)\frac{dR_n}{dr}+\left[\omega^2\left(1+\frac{2}{r}+\ldots\right)-\frac{n(n+1)}{r^2}\left(1+\frac{1}{r}+\ldots\right)\right]R_n$$

$$0=\frac{d^2\overset{o}{R}_n}{dr^2}+\left(\frac{1}{r^2}+\ldots\right)\frac{d\overset{o}{R}_n}{dr}+\left[\omega^2\left(1+\frac{2}{r}+\ldots\right)-\frac{n(n+1)}{r^2}\left(1+\frac{1}{r}+\ldots\right)-\frac{1}{r^3}\right]\overset{o}{R}_n \tag{6.8}$$

and their asymptotic forms

$$\begin{aligned} R_n &\approx e^{-i\omega v}-e^{i\omega v}\,e^{-i\omega\Delta_2 S} && a^2>27/4\\ &\ e^{-i\omega v} && a^2<27/4\\ \overset{o}{R}_n &\approx e^{-i\omega v}-e^{i\omega v}(-1)^n\,e^{-i\omega(1-2\ln 2\omega)}\,e^{2i\sigma_n} \end{aligned} \tag{6.9}$$

Inspection of these formulas shows that $\overset{o}{R}_n$ is rather good approximation of R_n: for large r both functions differ only in the phase shifts of the outgoing waves due to differences in the coefficients of their differential equations of order r^{-2} or less.

The scattered wave

$$\overset{s}{P}(r,\vartheta)\approx -e^{i\omega v}\sum_{n=1}^{\infty} D_nP_n^1(\cos\vartheta)\left[e^{-i\omega\Delta_2 S}-(-1)^n\,e^{2i\sigma_n}\,e^{-i\omega}\,e^{2i\omega\ln 2\omega}\right] \tag{6.10}$$

therefore is purely outgoing for large r. A detailed investigation of the phase shifts $\Delta_2 S$ and σ_n shows that for large n the square bracket in (6.10) gives contributions of order n^{-1} only, so that practically we only have to deal with finite sums.

7. *Concluding Remarks*

The splitting (6.7) of the wave into the incident part $\overset{o}{P}$ and the scattered part p is of course not unambiguous, but our choice seems to be natural. One should not be misled by the name 'incident' wave: substantially it is the wave changed by the potential r^{-1}, and the usual deflection of light rays, which is linear in the gravitational constant, is included in it.

Our second remark concerns the validity of equation (6.7): it is the exact solution of our problem, valid for arbitrary values of ω and $r>1$, and $\overset{o}{R}_n$ and

D_n are exactly given by (5.8) and (6.3). Approximations enter in only if we use the WBK method to get the asymptotic form for the radial functions $R_n(r)$.

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References

Breuer, R. A. *et al.* (1973). *Physical Review*, **D7**, 1002.
Darwin, C. (1959). *Proceedings of the Royal Society*, **249**, 187.
Ford, K. W. *et al.* (1959). *Annals of Physics*, 7, 239.
Herlt, E., *Acta Physica Polonica*, to be published.
Hönl, H. *et al.* (1961). In S. Flügge, *Encyclopedia of Physics*, XXV/1, p. 547. Springer, Berlin-Göttingen-Heidelberg.
Matzner, R. A. (1968). *Journal of Mathematical Physics*, **9**, 163.
Messiah, A. (1961). *Quantum Mechanics*, p. 421. North Holland Publishing Co., Amsterdam.
Misner, C. W. *et al.* (1972). *Physical Review Letters*, **28**, 998.
Mo, T. C. and Papas, C. H. (1971). *Physical Review*, **D3**, 1708.
Price, R. (1972). *Physical Review*, **D5**, 2439.
Ruffini, R. *et al.* (1972). *Lettere al Nuovo Cimento*, Ser. 2, **3**, 213.
Stephani, H. and Herlt, E. (1973). *International Journal of Theoretical Physics*, Vol. 8, No. 4, p. 243.
Stephani, H. (1974). *Journal of Mathematical Physics*, **15**, 14.